ISSN: 2582-8118 *Volume 3, Issue 1; Jan 2023*

INTERNATIONAL JOURNAL OF RESEARCH AND ANALYSIS IN SCIENCE AND ENGINEERING

Web: https://www.iarj.in/index.php/ijrase/index

3. Inequalities for Eigenvalues of Polynomial Operator of Elliptic Operator in Divergence Form on Metric Measure Space

Qinglin Liu

School of Mathematics and Statistics,
Nanjing University of Science and Technology;
Nanjing, Jiangsu, China.

ABSTRACT

In this paper, we prove some inequalities for eigenvalues of polynomial operator of elliptic operator in divergence form on a metric measure space isometrically immersed into an Euclidean space. As applications of the result, we also give some corresponding results on complete minimal submanifolds in Euclidean spaces and unit ball.

KEYWORDS

Eigenvalue, Inequality, Elliptic operator, Metric measure space.

1. Introduction:

The problem of eigenvalue estimation of differential operators is an important research direction in differential geometry. In recent years, the metric measure space has received more and more attention, and the study of eigenvalues of elliptic operators on it has become a frontier research problem internationally.

Let $(M, g, e^{-f}dv)$ be a smooth metric measure space and Ω be a bounded connected domain with smooth boundary $\partial\Omega$, where f is a smooth real-valued function, $\langle,\rangle$ is the Riemannian metric and dv is the Riemannian volume element on the Riemannian manifold $(M,\langle,\rangle)$. we consider the following eigenvalue problem:

$$\begin{cases} \mathrm{L}^2 u + p\mathrm{L}\, u + qu = \lambda\rho u, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \vec{\nu}}|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where p is a constant on Ω, q and ρ are two positive continuous functions on Ω, $\vec{v}$ denotes the outward unit normal to the boundary $\partial\Omega$, A is a symmetric positive definite (1,1)-tensor and the elliptic operator in divergences form L as follows:

$$\mathsf{L} = \mathrm{div}_f(A\nabla) = \mathrm{div}(A\nabla\cdot) - \langle \nabla f, A\nabla\cdot\rangle. \tag{1.2}$$

When M is compact and A is divergence-free($\mathrm{div}A = 0$), hence the operator L is a first-order perturbation of the Cheng-Yau[1] operator $\mathrm{div}(A\nabla)$. If the smooth function f is constant, then the elliptic operator L is the Cheng-Yau operator. The operator $\mathrm{div}(A\nabla)$ has been studied extensively in recent years. More results, we refer to [2–5].

Let $(e_1, e_2, \ldots, e_m)$ be a local orthonormal geodesic frame of TM, we can define a generalized mean curvature vector associated by the following formula:

$$\mathbf{H}_A = \frac{1}{n}\sum_{i=1}^{n} □(A(e_i), e_i) := \frac{1}{n}\mathrm{tr}(□ \circ A), \tag{1.3}$$

where □ be the second fundamental form on a smooth metric measure space $(M, g, e^{-f}dv)$ isometrically immersed in Euclidean Space $□^m$. More details and applications of $\mathbf{H}_A$ can be found in [6] and [7]. On this basis, Gomes[8] and others[9-11] have obtained some universal inequalities for eigenvalue of the operator L on different spaces.

The problem (1.1) has a real and discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to +\infty$, where each eigenvalue is repeated according to its multiplicity. In this paper, we can obtain some universal inequalities for the lower order eigenvalues of the elliptic operator L on smooth metric measure space.

Theorem 1.1 Let Ω be a bounded domain in an n-dimensional complete smooth metric measure space $(M, g, e^{-f}dv)$. Denote by λ_i the i-th eigenvalue of problem (1.1). Assume that $\varepsilon_1 I \le A \le \varepsilon_2 I$, $\rho_1 \le \rho \le \rho_2$, $q \le q_0$, where ε_2, ρ_1, ρ_2 and q_0 are positive constants. Set $H_0 = \sup_\Omega |\mathbf{H}_\mathbf{A}|$, $f_0 = \sup_\Omega |\nabla f|$, $A_0 = \sup_\Omega |\mathrm{tr}(\nabla A)|$, then

$$\sum_{i=1}^{n}(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}},, \frac{\rho_2}{\varepsilon_1 \rho_1^{\frac{3}{2}}}\left((A_0 + \varepsilon_2 f_0)^2 + n^2 H_0^2 + 2\sqrt{2}(A_0 + \varepsilon_2 f_0)E_1^{\frac{1}{2}} + 2E_1\right)^{\frac{1}{2}}$$
$$\times\left((A_0 + \varepsilon_2 f_0)^2 + n^2 H_0^2 + 2\sqrt{2}(A_0 + \varepsilon_2 f_0)E_1^{\frac{1}{2}} + 2E_i + 2nE_1 - 2\frac{\rho_1}{\rho_2} pn\varepsilon_1\right)^{\frac{1}{2}}, \tag{1.4}$$

where $E_1 = -p + \sqrt{p^2 - 4\rho_1(\frac{q_0}{\rho_2} - \lambda_1)}$.

Remark 1.1 Taking $p = 0, q = 0, \rho = 1$ in Theorem 1.1, then $\rho_1 = \rho_2 = 1, E_i = 2\lambda_i^{\frac{1}{2}}$. So (1.4) becomes the inequality (1.10) of Theorem 1.1 in Article [10].

Corollary 1.1 Set $W_0 = \max\left\{\sup_{\bar{\Omega}}\left|W_{e_k}\right| : k = n+1, \ldots, m\right\}$. Under the same assumptions as Theorem 1.1, we have

$$\sum_{i=1}^{n}(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \leq \frac{\rho_2}{\varepsilon_1 \rho_1^{\frac{3}{2}}}\left((A_0 + \varepsilon_2 f_0)^2 + (m-n)^2 \varepsilon_2^2 W_0^2 + 2\sqrt{2}\left(A_0 + \varepsilon_2 f_0\right) E_1^{\frac{1}{2}} + 2E_1\right)^{\frac{1}{2}}$$
$$\times\left((A_0 + \varepsilon_2 f_0)^2 + (m-n)^2 \varepsilon_2^2 W_0^2 + 2\sqrt{2}\left(A_0 + \varepsilon_2 f_0\right) E_1^{\frac{1}{2}} + (2+2n)E_1 - 2\frac{\rho_1}{\rho_2} pn\varepsilon_1\right)^{\frac{1}{2}}. \tag{1.5}$$

As we all know, when M is an n-dimensional Euclidean space, $H_0 = 0$. Thus we obtain the following corollary from Theorem 1.1.

Corollary 1.2 Let Ω be a connected bounded domain in an n-dimensional complete minimal submanifold in a Euclidean space. Denote by λ_i the i-th eigenvalue of problem (1.1). Then we have

$$\sum_{i=1}^{n}(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \leq \frac{\rho_2}{\varepsilon_1 \rho_1^{\frac{3}{2}}}\left((A_0 + \varepsilon_2 f_0)^2 + 2\sqrt{2}\left(A_0 + \varepsilon_2 f_0\right) E_1^{\frac{1}{2}} + 2E_i\right)^{\frac{1}{2}}$$
$$\times\left((A_0 + \varepsilon_2 f_0)^2 + 2\sqrt{2}\left(A_0 + \varepsilon_2 f_0\right) E_1^{\frac{1}{2}} + 2E_1 + 2nE_1 - 2\frac{\rho_1}{\rho_2} pn\varepsilon_1\right)^{\frac{1}{2}}. \tag{1.6}$$

Moreover, when M is an n-dimensional unit sphere, $H_0 \le \varepsilon_2$. Hence we have the following result:

Corollary 1.3 Let Ω be a connected bounded domain in an n-dimensional unit sphere $S^n(1)$. Denote by λ_i the i-th eigenvalue of problem (1.1). Then we have

$$\sum_{i=1}^{n}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}} \leq \frac{\rho_2}{\varepsilon_1\rho_1^{\frac{3}{2}}}\left((A_0+\varepsilon_2 f_0)^2+n^2\varepsilon_2^2+2\sqrt{2}\left(A_0+\varepsilon_2 f_0\right)E_1^{\frac{1}{2}}+2E_1\right)^{\frac{1}{2}}$$
$$\times\left((A_0+\varepsilon_2 f_0)^2+n^2\varepsilon_2^2+2\sqrt{2}\left(A_0+\varepsilon_2 f_0\right)E_1^{\frac{1}{2}}+2E_1+2nE_1-2\frac{\rho_1}{\rho_2}pn\varepsilon_1\right)^{\frac{1}{2}}. \tag{1.7}$$

2. Proofs of the main results:

Before giving the result, we give some necessary lemmas that will play an important role in the proof of Theorem 1.1.

Lemma 2.1 Let u_i be the orthonormal eigenfunction corresponding to the i-th eigenvalue λ_i of problem (1.1). For any function $g_i \in C^2(\bar{\Omega})$, if it satisfies $\int_\Omega g_i u_1 = 0$, then we have

$$\int_\Omega u_1 \mathrm{Y}_i = 0 \tag{2.1}$$

and

$$\int_\Omega g_i u_1 \mathrm{Y}_i = \int_\Omega 2\left(\left\langle \nabla g_i, A\nabla u_1\right\rangle + \frac{u_1\mathcal{L}g_i}{2}\right)^2 - 2u_1\mathcal{L}u_1\left\langle \nabla g_i, A\nabla g_i\right\rangle - pu_1^2\left\langle \nabla g_i, A\nabla g_i\right\rangle, \tag{2.2}$$

where

$$\mathrm{Y}_i = 2\left\langle \nabla g_i, A\nabla(\mathcal{L}u_1)\right\rangle + 2\mathcal{L}\left\langle \nabla g_i, A\nabla u_1\right\rangle + \mathcal{L}(u_1\mathcal{L}g_i) + \mathcal{L}u_1\mathcal{L}g_i + 2p\left\langle \nabla g_i, A\nabla u_1\right\rangle + pu_1\mathcal{L}g_i.$$

Proof Using the divergence theorem, we can get the following equations:

$$\begin{aligned}
&\int_\Omega u_1\mathcal{L}\langle \nabla g_i, A\nabla u_1\rangle + \int_\Omega u_1\langle \nabla g_i, A\nabla(\mathcal{L}u_1)\rangle \\
&= \int_\Omega \mathcal{L}u_1\langle \nabla g_i, A\nabla u_1\rangle - \int_\Omega \mathcal{L}u_1\langle \nabla g_i, A\nabla u_1\rangle - \int_\Omega u_1\mathcal{L}u_1\mathcal{L}g_i \\
&= -\int_\Omega u_1\mathcal{L}u_1\mathcal{L}g_i
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\int_\Omega u_1^2 \mathcal{L} g_i + 2\int_\Omega u_1 \langle \nabla g_i, A\nabla u_1\rangle \\ &= \int_\Omega (g_i u_1 \mathcal{L} u_1 + g_i u_1 \mathcal{L} u_1 + 2 g_i \langle \nabla u_1, A\nabla u_1\rangle) - \int_\Omega (2 g_i u_1 \mathcal{L} u_1 + 2 g_i \langle \nabla u_1, A\nabla u_1\rangle) \\ &= 0.\end{aligned} \tag{2.4}$$

Since the operator $\mathcal{L}$ is self-adjoint, then

$$\int_\Omega u_1 \mathcal{L} u_1 \mathcal{L} g_i = \int_\Omega u_1 \mathcal{L}(u_1 \mathcal{L} g_i). \tag{2.5}$$

From (2.3-2.5), we get

$$\begin{aligned}\int_\Omega \Upsilon_i u_1 = \int_\Omega \big[& u_1 \mathcal{L} g_i \mathcal{L} u_1 + u_1 \mathcal{L} g_i \mathcal{L} u_1 + 2\mathcal{L} u_1 \langle \nabla g_i, A\nabla u_1\rangle - 2\mathcal{L} u_1 \langle \nabla g_i, A\nabla u_1\rangle \\ & -2u_1 \mathcal{L} u_1 \mathcal{L} g_i + p u_1^2 \mathcal{L} g_i + 2 p u_1 \langle \nabla g_i, A\nabla u_1\rangle \big] = 0.\end{aligned} \tag{2.6}$$

This completes the proof of (2.1).

Moreover, we have

$$2\int_\Omega g_i u_1 \langle \nabla u_1, A\nabla(\mathcal{L} g_i)\rangle = \int_\Omega \left(2u_1 \mathcal{L} g_i \langle \nabla u_1, A\nabla g_i\rangle + u_1^2 (\mathcal{L} g_i)^2 - g_i u_1^2 \mathcal{L}^2 g_i\right), \tag{2.7}$$

$$\int_\Omega g_i u_1 \mathcal{L} \langle \nabla g_i, A\nabla u_1\rangle = \int_\Omega \mathcal{L} g_i u_1 \langle \nabla g_i, A\nabla u_1\rangle + 2\langle \nabla g_i, A\nabla u_1\rangle^2 + g_i \mathcal{L} u_1 \langle \nabla g_i, A\nabla u_1\rangle \tag{2.8}$$

and

$$\int_\Omega g_i u_1 \langle \nabla g_i, A\nabla(\mathcal{L} u_1)\rangle = -\int_\Omega u_1 \mathcal{L} u_1 \langle \nabla g_i, A\nabla g_i\rangle + g_i \mathcal{L} u_1 \langle \nabla g_i, A\nabla u_1\rangle + g_i u_1 \mathcal{L} g_i \mathcal{L} u_1. \tag{2.9}$$

Substituting (2.7-2.9) into the following equation, we have

$$\begin{aligned}&\int_\Omega g_i u_1 \Upsilon_i \\ &= \int_\Omega 2 g_i u_1 \langle \nabla g_i, A\nabla(\mathcal{L} u_1)\rangle + 2 g_i u_1 \mathcal{L} \langle \nabla g_i, A\nabla u_1\rangle + g_i u_1 \mathcal{L}(u_1 \mathcal{L} g_i) \\ &\quad + g_i u_1 \mathcal{L} u_1 \mathcal{L} g_i + 2 p g_i u_1 \langle \nabla g_i, A\nabla u_1\rangle + p g_i u_1^2 \mathcal{L} g_i \\ &= \int_\Omega 2\left(\langle \nabla g_i, A\nabla u_1\rangle + \frac{u_1 \mathcal{L} g_i}{2}\right)^2 - 2u_1 \mathcal{L} u_1 \langle \nabla g_i, A\nabla g_i\rangle - p u_1^2 \langle \nabla g_i, A\nabla g_i\rangle.\end{aligned} \tag{2.10}$$

The proof of Lemma 2.1 is complete.

Lemma 2.2 Let u_i be the orthonormal eigenfunction corresponding to the i-th eigenvalue λ_i of problem (1.1). For any function $g_i \in C^2(\bar{\Omega})$, if it satisfies $\int_\Omega g_i u_1 = 0$, then we have

$$\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}} \int_\Omega u_1^2 \left(\nabla g_i, A\nabla g_i\right) \le \delta \int_\Omega \Psi_i + \frac{1}{\delta}\int_\Omega \frac{1}{\rho}\left(\frac{u_1 \mathsf{L} g_i}{2} + \langle \nabla g_i, A\nabla u_1\rangle\right)^2, \quad (2.11)$$

where δ is any positive constant and

$$\Psi_i = 2\left(\left\langle \nabla g_i, A\nabla u_1\right\rangle + \frac{u_1 \mathsf{L} g_i}{2}\right)^2 - 2u_1 \mathsf{L} u_1 \left\langle \nabla g_i, A\nabla g_i\right\rangle - p u_1^2 \left\langle \nabla g_i, A\nabla g_i\right\rangle.$$

Proof Consider the test functions $\varphi_i = g_i u_1 - a_i u_1$, where $a_i = \int_\Omega \rho g_i u_1^2$. It is easy to check that $\int_\Omega \rho \varphi_i u_1 = 0$. Moreover, we have

$$\varphi_i |_{\partial\Omega} = \frac{\partial \varphi_i}{\partial \nu} |_{\partial\Omega} = 0 \quad \text{and} \quad \int_\Omega \rho \varphi_i u_j + 1 = 0, \quad \forall 0 < j \le i.$$

Therefore, using the Rayleigh-Ritz inequality, we get

$$\lambda_{i+1} \int_\Omega \rho \varphi_i^2 \le \int_\Omega \varphi_i \left[\mathsf{L}^2 \varphi_i + p\mathsf{L}\,\varphi_i + q\varphi_i\right] = \int_\Omega \varphi_i \left[\mathsf{L}^2\left(g_i u_1\right) + p\mathsf{L}\left(g_i u_1\right) + q\left(g_i u_1\right)\right]. \quad (2.12)$$

It is easy to obtain that

$$\mathsf{L}^2(g_i u_1) = g_i \mathsf{L}^2 u_1 + 2\left\langle \nabla g_i, A\nabla(\mathsf{L}\, u_1)\right\rangle + 2\mathsf{L}\left\langle \nabla g_i, A\nabla u_1\right\rangle + \mathsf{L}\,(u_1 \mathsf{L} g_i) + \mathsf{L}\, u_1 \mathsf{L} g_i. \quad (2.13)$$

Substituting (2.13) into (2.12), we have

$$(\lambda_{i+1} - \lambda_1)\int_\Omega \rho \varphi_i^2 \le \int_\Omega g_i \Upsilon_i u_1 - a_i \int_\Omega \Upsilon_i u_1. \quad (2.14)$$

Substituting the equation (2.1) and (2.2) in Lemma 2.1 into (2.14), we get

$$\left(\lambda_{k+1} - \lambda_i\right)\int_\Omega \rho \varphi_i^2 \le \int_\Omega \Psi_i. \quad (2.15)$$

Notice that $\int_\Omega u_1\left(\langle\nabla g_i, A\nabla u_1\rangle + \frac{u_1\mathsf{L}g_i}{2}\right) = 0.$ Hence, from the divergence Theorem, we have

$$-2\int_\Omega \varphi_i\left(\langle\nabla g_i, A\nabla u_1\rangle + \frac{u_1\mathsf{L}g_i}{2}\right) = \int_\Omega\left[g_i u_1^2\mathsf{L}g_i + g_i\langle\nabla g_i, A\nabla u_1^2\rangle\right] = \int_\Omega u_1^2\langle\nabla g_i, A\nabla g_i\rangle. \tag{2.16}$$

Multiplying both sides of (2.16) by $(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}$, we deduce

$$\begin{aligned}
&(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\left(\int_\Omega u_1^2\langle\nabla g_i, A\nabla g_i\rangle\right)\\
&\leq \delta(\lambda_{i+1}-\lambda_i)\int_\Omega \rho\varphi_i^2 + \frac{1}{\delta}\int_\Omega\left[\frac{1}{\sqrt{\rho}}\left(\langle\nabla g_i, A\nabla u_1\rangle + \frac{u_1\mathsf{L}g_i}{2}\right)\right]^2 \\
&\leq \delta\int_\Omega \Psi_i + \frac{1}{\delta}\int_\Omega\frac{1}{\rho}\left(\langle\nabla g_i, A\nabla u_1\rangle + \frac{u_1\mathsf{L}g_i}{2}\right)^2.
\end{aligned} \tag{2.17}$$

The proof of Lemma 2.2 is completed.

Now we give the proof of Theorems 1.1.

proof of Theorems 1.1 Let $y=(y_1, y_2, \ldots, y_m)$ be the position vector of the immersion of M on $\square^{m}$. Taking $g_i = y_i$ in lemma 2.2, and summing over i from 1 to m, and set $\ell_i = \left(\langle\nabla y_i, A\nabla u_1\rangle + \frac{u_1\mathsf{L}y_i}{2}\right)$, we get

$$\begin{aligned}
&\sum_{i=1}^{m}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\left(\int_\Omega u_1^2\langle\nabla y_i, A\nabla y_i\rangle\right) \leq \sum_{i=1}^{m}\frac{1}{\delta}\int_\Omega\frac{1}{\rho}\ell_i^2\\
&+\delta\sum_{i=1}^{m}\int_\Omega\left[2\ell_i^2 - 2\left(u_1\mathsf{L}u_1\langle\nabla y_i, A\nabla y_i\rangle\right) - pu_1^2\langle\nabla y_i, A\nabla y_i\rangle\right].
\end{aligned} \tag{2.18}$$

According the Schwarz inequality, there is

$$\int_\Omega u_1\mathsf{L}u_1 \;,,\; \left(\int_\Omega u_1^2\int_\Omega(\mathsf{L}u_1)^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{\rho_1}\int_\Omega u_1\mathsf{L}^2u_1\right)^{\frac{1}{2}}. \tag{2.19}$$

Hence

$$\lambda_1 .. \rho_1 \left(\int_\Omega u_1 \mathrm{L} u_1 \right)^2 + p \int_\Omega u_1 \mathrm{L} u_1 + \frac{q_0}{\rho_2}. \tag{2.20}$$

Solving this quadratic equation, we get

$$\int u_1 \mathrm{L} u_1 \leq \frac{E_1}{2\rho_1}, \tag{2.21}$$

where $E_1 = -p + \sqrt{p^2 - 4\rho_1 (\frac{q_0}{\rho_2} - \lambda_1)}$.

From the article by Gomes and Miranda[8], we can give some results as follows

$$\sum_{i=1}^{m} \left(\nabla y_i, A \nabla y_i \right) = \mathrm{tr}(A), \tag{2.22}$$

$$\sum_{i=1}^{m} \left| \left\langle \nabla y_i, A \nabla u_1 \right\rangle \right|^2 = \left| A \nabla u_1 \right|^2, \tag{2.23}$$

$$\sum_{i=1}^{m} \left(\mathrm{L} y_i \right)^2 = n^2 \mathbf{H_A}^2 + \left\langle \mathrm{tr}(\nabla A), \mathrm{tr}(\nabla A) - 2A \nabla f \right\rangle + \left| A \nabla f \right|^2 \tag{2.24}$$

and

$$\sum_{i=1}^{m} \mathrm{L} y_i \langle \nabla y_i, A \nabla u_1 \rangle = \langle \mathrm{tr}(\nabla A), A \nabla u_1 \rangle - \langle A(\nabla f), A \nabla u_1 \rangle. \tag{2.25}$$

After computation, some inequalities can be obtained as follows:

$$\sum_{i=1}^{m} \int_\Omega u_1^2 \left(\mathrm{L} y_i \right)^2 \leq \frac{A_0^2}{\rho_1} + \frac{2}{\rho_1} \varepsilon_2 A_0 f_0 + \frac{1}{\rho_1} \varepsilon_2^2 f_0^2 + \frac{n^2}{\rho_1} \left| H_0 \right|^2, \tag{2.26}$$

$$\sum_{i=1}^{m} \int_\Omega u_1 \mathrm{L} y_i \left\langle \nabla y_i, A \nabla u_1 \right\rangle ,, \frac{A_0 + \varepsilon_2 f_0}{\rho_1} (\frac{\varepsilon_2 E_1}{2})^{\frac{1}{2}}, \tag{2.27}$$

$$\sum_{i=1}^{m} \int_\Omega \left\langle \nabla y_i, A \nabla u_1 \right\rangle^2 = \int_\Omega \left| A(\nabla u_1) \right|^2 = \int_\Omega \left| \left\langle A(\nabla u_1), A(\nabla u_1) \right\rangle \right| \leq \frac{\varepsilon_2 E_1}{2\rho_1}, \tag{2.28}$$

$$-\sum_{i=1}^{m}\int_{\Omega} u_1 \mathrm{L} u_1 \left\langle \nabla y_i, A\nabla y_i \right\rangle = -\int u_1 \mathrm{L} u_1 \,\mathrm{tr}(A) \leq \frac{n\varepsilon_2 E_1}{2\rho_1} \tag{2.29}$$

and

$$-p\sum_{i=1}^{m}\int_{\Omega} u_1^2 \left\langle \nabla y_i, A\nabla y_i \right\rangle = -p\int u_1^2 \,\mathrm{tr}(A) \leq -\frac{p}{\rho_2} n\varepsilon_1. \tag{2.30}$$

In addition, after a simple computation we can obtain

$$\begin{aligned}
&\sum_{i=1}^{m}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\int u_1^2 \left\langle \nabla y_i, A\nabla y_i \right\rangle \\
&\geq \frac{\varepsilon_1}{\rho_2}\sum_{i=1}^{m}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\left|\nabla y_i\right|^2 \\
&\geq \frac{\varepsilon_1}{\rho_2}\sum_{i=1}^{n}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\left|\nabla y_i\right|^2 + \left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\sum_{\beta=n+1}^{m}\left|\nabla h_\beta\right|^2 \\
&= \frac{\varepsilon_1}{\rho_2}\sum_{i=1}^{n}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\left|\nabla y_i\right|^2 + \left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\left(n-\sum_{i=1}^{n}\left|\nabla y_i\right|^2\right) \\
&\geq \sum_{i=1}^{n}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\left|\nabla y_i\right|^2 + \sum_{i=1}^{n}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}\left(1-\left|\nabla y_i\right|^2\right) \\
&= \frac{\varepsilon_1}{\rho_2}\sum_{i=1}^{n}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.31}$$

Substituting (2.26-2.31) into (2.18), we get

$$\begin{aligned}
&\frac{\varepsilon_1}{\rho_2}\sum_{i=1}^{n}\left(\lambda_{i+1}-\lambda_1\right)^{\frac{1}{2}} \leq \delta\left[\frac{1}{2\rho_1}\left(2nE_i - 2\frac{\rho_1}{\rho_2}pn\varepsilon_1\right)\right] \\
&+\left[\frac{1}{2\rho_1}\left(\delta+\frac{1}{\delta\rho_1}\right)\left(A_0^2 + 2\varepsilon_2 A_0 f_0 + \varepsilon_2^2 f_0^2 + nH_0^2 + 2\sqrt{2}\left(A_0+\varepsilon_2 f_0\right)E_i^{\frac{1}{2}} + 2E_i\right)\right].
\end{aligned} \tag{2.32}$$

Taking

$$\delta = \left[\frac{\left(A_0+\varepsilon_2 f_0\right)^2 + nH_0^2 + 2\sqrt{2}\left(A_0+\varepsilon_2 f_0\right)E_1^{\frac{1}{2}} + 2E_1}{\rho_1[\left(A_0+\varepsilon_2 f_0\right)^2 + nH_0^2 + 2\sqrt{2}\left(A_0+\varepsilon_2 f_0\right)E_1^{\frac{1}{2}} + (2+2n)E_1 - 2\frac{\rho_2}{\rho_1}pn\varepsilon_1]}\right]^{\frac{1}{2}}$$

in (2.32), we can get (1.4). The proof of Theorem 1.1 is completed.

Proof of Corollary 1.1 Let W_{e_i} be the Weingarten operator of the immersion with respect to e_i, set $W_0 = \max\left\{\sup_{\bar{\Omega}} \left|W_{e_k}\right| : k = n+1, \ldots, m\right\}$. Then we have

$$
\begin{aligned}
&\| \mathrm{tr}(\Box \circ A) \|^2 \\
&= \left\| \sum_{i=1}^{k} \Box\left(A(e_i), e_i \right) \right\|^2 = \left\| \sum_{i=1}^{k} \sum_{k=n+1}^{m} \left\langle \Box\left(Ae_i, e_i \right), e_k \right\rangle e_k \right\|^2 \\
&= \left\| \sum_{k=n+1}^{m} \left(\sum_{i=1}^{k} \left\langle W_{e_k} e_i, Ae_i \right\rangle \right) e_k \right\|^2 = \left\| \sum_{k=n+1}^{m} \left\langle W_{e_k}, A \right\rangle e_k \right\|^2 \\
&\leq \sum_{k=n+1}^{m} \left| \left\langle W_{e_k}, A \right\rangle \right|^2 \sum_{k=n+1}^{m} \left| e_k \right|^2 \leq (m-n)^2 W_0^2 \varepsilon_2^2.
\end{aligned}
$$

This finished the proof of Corollary 1.1.

3. References:

1. S. Y. CHENG, S. T. YAU. Hypersurfaces with constant scalar curvature. Mathematische Annalen, 225: 195-204, 1977.
2. M. P. DO CARMO, Q. L. WANG and C. Y. XIA. Inequalities for eigenvalues of elliptic operators in divergence form on Riemannian manifolds. Annali di Matematica Pura ed Applicata, 189: 643-660, 2010.
3. F. DU, C. X. WU and G. H. LI. Estimates for eigenvalues of fourth-order elliptic operators in divergence form. Bulletin of the Brazilian Mathematical Society New, 46(3): 437-459, 2015.
4. S. AZAMI. Inequalities for eigenvalues of fourth order elliptic operators in divergence form on Riemannian manifolds. Journal of Mathematical Analysis and Applications, 481: 123492, 2020.
5. H. J. SUN and D. G. CHEN. Inequalities for lower order eigenvalues of second order elliptic operators in divergence form on Riemannian manifolds. Archiv der Mathematik, 101: 381-393, 2013.
6. J. F. GROSJEAN. Extrinsic upper bounds for the first eigenvalue of elliptic operators. Hokkaido Math, 33(2): 319–339, 2004.
7. J. ROTH. Reilly-type inequalities for Paneitz and Steklov eigenvalues. Potential Anal, 53(3): 773–798, 2020.
8. J. N. V. GOMES and J. F. R. MIRANDA. Eigenvalue estimates for a class of elliptic differential operators in divergence form. Nonlinear Analysis, 176: 1-9, 2018.
9. Y. ZHU, G. LIU and F. DU. Eigenvalue inequalities of elliptic operators in weighted divergence form on smooth metric measure spaces. J Inequal Appl, 2016: 1-15, 2016.
10. F. M. C. ARAÚJO. Inequalities for eigenvalues of fourth-order elliptic operators in divergence form on complete Riemannian manifolds. Zeitschrift für angewandte Mathematik und Physik, 73: 47, 2022.

11. Z. X. WANG. Eigenvalue estimates for a class of elliptic operators in divergence form. Zhengzhou University, 2019.