ISSN: 2582-8118 ***Volume 3, Issue 1; Jan 2023***

INTERNATIONAL JOURNAL OF RESEARCH AND ANALYSIS IN SCIENCE AND ENGINEERING

Web: https://www.iarj.in/index.php/ijrase/index

2. An Inverse Problem for Strum-Liouville Operators with A Delay and The Eigenparameter Boundary Condition

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ABSTRACT

In this work we consider the boundary value problems for Sturm-Liouville operators with a constant delay $\tau \in \left[\frac{2\pi}{5}, \frac{\pi}{2}\right)$ *under eigenparameter boundary condition. Under some assumptions, the uniqueness of the inverse specteal problems is proved, where the potential, parameters in boundary conditions and the delay are uniquely determined by two spectra of the different boundary conditions.*

KEYWORDS

Sturm-Liouville operators, Eigenparameter, Constant delay, Inverse problem.

1. Introduction:

In this paper we consider the Sturm-Liouville boundary value problems $L_j\left(j=1,2\right)$:

$$-y''(x)+q(x)y(x-\tau)=\lambda y(x), \quad x\in(0,\pi) \tag{1.1}$$

with boundary conditions

$$y(0)=0, \tag{1.2}$$

$$y'(\pi)+H_j(\lambda)y(\pi)=0, \tag{1.3}$$

where $H_j(\lambda) = a_j\lambda + b_j, \lambda$ is the spectral parameter, $\tau \in \left[\frac{2\pi}{5}, \frac{\pi}{2}\right)$, the complex-valued potential $q(x)$ is continuous in $[0,\pi], q(x) = 0$ for $x \in (0,\tau)$. Moreover, $a_1a_2 \neq 0, b_1 \neq b_2$ and $H(m) := H_2(m^2) - H_1(m^2) \neq 0$.

Recently, differential operators with constant delays have attracted more and more attention of researchers because they are widely used in engineering and natural sciences (e.g., see the monographs [13, 18] and the references therein). Inverse spectral problems of the differential operators consist in recovering operators from the given spectral characteristics. The research contents involve the existence, uniqueness and reconstruction of Sturm-Liouville operators.

Comparing with the inverse spectral theroy of classical differential operators (see [12] and the references therein), it is more difficult to study the inverse problems of differential operators with constant delays. This is because the main methods of the inverse problems theory are not applicable for them. Therefore, there are only isolated results in this direction and do not form a complete picture. For example, in [1-9,11,20,23-24] they provided a few results of the inverse problems of Sturm-Liouville operators with a constant delay on a finite interval.

In addition, as for the above papers (see [1-9, 11, 20, 23-24]), we note that the characteristic functions depend linearly on the potential in the case of large delay when $\tau \geq \frac{\pi}{2}$, i.e., the inverse problem becomes linear (see [3.23]). For $\tau < \frac{\pi}{2}$, this nonlinear case is essentially more difficult for investigating and constructing the solution of the inverse problems. The characteristic functions depend nonlinearly on the potential, i.e., the inverse problem becomes nonlinear (see [4,20]).

In the papers [10,14,16-17,19], authors studied the inverse problems for Sturm-Liouville operators with eigenparameter boundary conditions. Moreover, we also note that there are some researches on the operators with one constant delay under eigenparameter boundary conditions (see [15,21]). In [21], authors studied two boundary value problems (1.4), (1.5), (1.6) and (1.4), (1.5), (1.7) for $\tau \in \left[\frac{2\pi}{5}, \pi\right)$:

$$-y''(x) + q(x)y(x-\tau) = \lambda y(x), \quad x \in (0,\pi) \tag{1.4}$$

$$y(0) = 0, \tag{1.5}$$

$$y(\pi) = 0, \tag{1.6}$$

$$y'(\pi)+P(\lambda)y(\pi)=0. \qquad (1.7)$$

Function $P(\lambda)$ is normalized polynomial with degree $s, s\in\mathrm{N}$, and complex coefficients. The authors proved uniqueness and gave procedure for constructing potential. In the first case, for $\tau\in\left[\frac{\pi}{2},\pi\right)$, they showed that Fourier coefficients of a potential are uniquely determined by two spectra. In the second case for $\tau\in\left[\frac{2\pi}{5},\frac{\pi}{2}\right)$, they constructed integral equation about potential and they proved that this integral equation has a unique solution. Also, they showed that other parameters are uniquely determined by two spectra.

In this paper we consider the inverse problems of Sturm-Liouville operators for $\tau\in\left[\frac{2\pi}{5},\frac{\pi}{2}\right)$ under Dirichlet/linear and Dirichlet/linear boundary conditions. In the case of $\tau\geq\frac{\pi}{2}$, it has been studied in [22], where we proved uniqueness and gave procedure for constructing potential under the conditions $\pi-\tau\in\mathrm{Q}$. However, this case may be not true as soon as $\tau\leq\frac{2\pi}{5}$. It needs to be further studied separately.

Moreover, we suppose that b_1, b_2, integral $I_1=\int_\tau^\pi q(t)dt\neq 0$ and $I_2=\int_{2\tau}^\pi\int_\tau^{t-\tau}q(t)$ $q(s)dsdt$ are known and $\pi-\tau\in\mathrm{Q}$. We will prove that the $H_j\ (j=1,2)$ and the potential $q(x)$ are uniquely determined from the spectra of $L_j,(j=1,2)$. To be more precise, let $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ be the eigenvalues of $L_j\ (j=1,2)$. The inverse problems are to determine potential $q(x), H_j$ and τ from $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}(j=1,2)$.

This paper is organized as follows. In Section 2 we study the spectra of the boundary value problems (1.1) -(1.3) and introduce transformation of characteristic functions, which is needed for constructing the integral equation with the potential. In Section 3 we consider the inverse spectral problems of recovering the potential $q(x)$ and other parameters, and prove that the integral equation has unique solution.

2. Properties of Spectral Characteristics:

Let $\lambda=\rho^2, \rho=s+it$ and the function $y(x,\lambda)$ be the solution of the equation (1.1) under initial conditions $y(0)=0, y'(0)=1$, then $y(x,\lambda)$ is the unique solution of the integral equation

$$y(x,\lambda)=\frac{\sin(\rho x)}{\rho}+\int_{\tau}^{x}\frac{q(t)\sin(\rho(x-t))}{\rho}y(t-\tau,\lambda)\,dt. \tag{2.1}$$

For $x\in[0,\tau)$, the solution of (2.1) is

$$y(x,\lambda)=\frac{\sin(\rho x)}{\rho}+\int_{\tau}^{x}\frac{q(t)\sin(\rho(x-t))}{\rho}y(t-\tau,\lambda)\,dt=\frac{\sin(\rho x)}{\rho}.$$

For $x\in(\tau,2\tau]$, the solution of (2.1) is

$$y(x,\lambda)=\frac{\sin(\rho x)}{\rho}+\frac{1}{\rho^2}\int_{\tau}^{x}q(t)\sin(\rho(x-t))\sin(\rho(t-\tau))\,dt.$$

For $x\in[2\tau,\pi]$ the solution is

$$\begin{aligned}y(x,\lambda)&=\frac{\sin(\rho x)}{\rho}+\frac{1}{\rho^2}\int_{\tau}^{x}q(t)\sin(\rho(x-t))\sin(\rho(t-\tau))\,dt+\\&\frac{1}{\rho^3}\int_{2\tau}^{x}\int_{\tau}^{t-\tau}q(t)q(s)\sin(\rho(x-t))\sin(\rho(t-\tau-s))\sin(\rho(s-\tau))\,dsdt.\end{aligned} \tag{2.2}$$

Moreover, we have

$$\begin{aligned}y'(x,\lambda)&=\cos(\rho x)+\frac{1}{\rho}\int_{\tau}^{x}q(t)\cos(\rho(x-t))\sin(\rho(t-\tau))\,dt+\\&\frac{1}{\rho^2}\int_{2\tau}^{x}\int_{\tau}^{t-\tau}q(t)q(s)\cos(\rho(x-t))\sin(\rho(t-\tau-s))\sin(\rho(s-\tau))\,dsdt.\end{aligned} \tag{2.3}$$

Let

$$\Delta_j(\lambda)=y'(\pi,\lambda)+H_j(\lambda)\,y(\pi,\lambda),\ j=1,2. \tag{2.4}$$

Using (2.2) and (2.3), we have

$$\begin{aligned}\Delta_j(\lambda) &= y'(\pi,\lambda)+H_j(\lambda)y(\pi,\lambda)\\ &= H_j(\lambda)\frac{\sin(\rho\pi)}{\rho}+\cos(\rho\pi)+\frac{1}{\rho}\int_\tau^x q(t)\cos(\rho(\pi-t))\sin(\rho(t-\tau))dt\\ &+\frac{1}{\rho^2}\int_{2\tau}^x\int_\tau^{t-\tau} q(t)q(s)\cos(\rho(\pi-t))\sin(\rho(t-\tau-s))\sin(\rho(s-\tau))dsdt\\ &+\frac{H_j(\lambda)}{\rho^2}\int_\tau^\pi q(t)\sin(\rho(\pi-t))\sin(\rho(t-\tau))dt\\ &+\frac{H_j(\lambda)}{\rho^3}\int_{2\tau}^\pi\int_\tau^{t-\tau} q(t)q(s)\sin(\rho(\pi-t))\sin(\rho(t-\tau-s))\sin(\rho(s-\tau))dsdt.\end{aligned} \tag{2.5}$$

It is easy to verify that $\Delta_j(\lambda)(j=1,2)$ is the characteristic function of $L_j(j=1,2)$, whose zeros coincide with the eigenvalues of $L_j(j=1,2)$. Now, using (2.5) by the well-known method ([12], Ch.1), we obtain zeros of $\Delta_j(\lambda)$

$$\rho_n^{(j)} = n+\frac{1+a_jI_1C_2(n,\tau)+a_jA(n,\tau)}{a_jn\pi}+O\left(\frac{1}{n^2}\right). \tag{2.6}$$

Where

$$\begin{aligned} I_1 &= \int_\tau^\pi q(t)dt,\\ A(\rho,\tau) &= \frac{1}{2}\int_\tau^\pi q(t)\cos(\rho(2t-\tau))dt,\\ C_1(\rho,\tau) &= \frac{1}{2}\sin(\rho\tau),\\ C_2(\rho,\tau) &= \frac{1}{2}\cos(\rho\tau).\end{aligned} \tag{2.7}$$

Moreover, $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ is the spectrum of $L_j(j=1,2), \lambda_n^{(j)}=\left(\rho_n^{(j)}\right)^2$. Since the $\Delta_j(\lambda), j=1,2$, are entire in λ in order $\frac{1}{2}$, by Hadamard's factorization theorem, the characteristic functions are uniquely determined by spectra of $L_j(j=1,2)$. The following lemma holds by the well-known method ([12], Ch.1).

Lemma 2.1. The characteristic function $\Delta_j(\lambda)(j=1,2)$, which is entire functions of λ of order $1/2$, can be uniquely determined by the specification of the spectrum $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}(j=1,2)$ and $a_j(j=1,2)$ by the formula

$$\Delta_j(\lambda) = a_j \pi \lambda \prod_{n=1}^{\infty} \frac{\lambda - \lambda_n^{(j)}}{n^2}. \tag{2.8}$$

3. Main Result:

In this section we prove the uniqueness of the solution. Firstly we give the following lemmas.

Lemma 3.1. The spectrum $\left\{\lambda_n^{(1)}\right\}_{n\geq 0}$ of the boundary spectral problem L_1 uniquely determines the delay τ.

Proof. Since there are infinitely many $k \in \mathrm{N}$ and $\delta > 0$ with property $\left|\sin(k\tau)\right| > \delta > 0$. From the assumption $I_1 \neq 0$, we have

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\lambda_{k-2}^{(1)} - (k-2)^2 - \lambda_{k+2}^{(1)} + (k+2)^2}{\lambda_{k-1}^{(1)} - (k-1)^2 - \lambda_{k+1}^{(1)} + (k+1)^2} \\
&= \lim_{k\to\infty} \frac{\dfrac{2}{a_1\pi} + \dfrac{2I_1 \cos\left((k-2)\tau\right)}{\pi} - \dfrac{2}{a_1\pi} - \dfrac{2I_1 \cos\left((k+2)\tau\right)}{\pi}}{\dfrac{2}{a_1\pi} + \dfrac{2I_1 \cos\left((k-1)\tau\right)}{\pi} - \dfrac{2}{a_1\pi} - \dfrac{2I_1 \cos\left((k+1)\tau\right)}{\pi}} \\
&= \lim_{k\to\infty} \frac{\cos\left((k-2)\tau\right) - \cos\left((k+2)\tau\right)}{\cos\left((k-1)\tau\right) - \cos\left((k+1)\tau\right)} \\
&= \lim_{k\to\infty} \frac{\sin(k\tau)\sin(2\tau)}{\sin(k\tau)\sin\tau} = 2\cos\tau.
\end{aligned}$$

Therefore, we obtain

$$\tau = \arccos\left[\frac{1}{2}\lim_{k\to\infty} \frac{\lambda_{k-2}^{(1)} - (k-2)^2 - \lambda_{k+2}^{(1)} + (k+2)^2}{\lambda_{k-1}^{(1)} - (k-1)^2 - \lambda_{k+1}^{(1)} + (k+1)^2}\right]. \tag{3.1}$$

Lemma 3.2. Let $\pi - \tau \in \mathrm{Q}$. The spectrum $\left\{\lambda_n^{(1)}\right\}_{n\geq 0}$ of the boundary spectral problem L_1 uniquely determines I_1.

Proof. From (2.5) we conclude

$$\Delta_1(\lambda) = a_1\rho\sin(\rho\pi) + \cos(\rho\pi) + \frac{a_1 I_1 \cos(\rho(\pi-\tau))}{2} + O\left(\frac{e^{|\mathrm{Im}\rho|\pi}}{\rho}\right). \quad (3.2)$$

Since $a_1 \neq 0$, from (2.8) and (3.2) we have

$$\pi\prod_{n=1}^{\infty}\frac{\lambda-\lambda_n^{(1)}}{n^2} = -\rho\sin(\rho\pi) + \frac{\cos(\rho\pi)}{a_1} + \frac{I_1\cos(\rho(\pi-\tau))}{2} + O\left(\frac{e^{|\mathrm{Im}\rho|\pi}}{\rho}\right).$$
(3.3)

By some calculation, it is easy to see that the zeros $\left\{\rho_n : \rho_n = n + \frac{1}{2}, n \in \mathrm{Z}\right\}$ of $\cos(\rho\pi)$ and the zeros $\left\{\rho_m : \rho_m = \frac{\left(m+\frac{1}{2}\right)\pi}{\pi-\tau}, m \in \mathrm{Z}\right\}$ of $\cos(\rho(\pi-\tau))$. It is easy to prove that the assumption $\pi - \tau \in \mathrm{Q}$ implies the functions $\cos(\rho\pi)$ and $\cos(\rho(\pi-\tau))$ do not have any common zeros. Denote

$$G_\delta := \left\{\lambda : \left|\rho - \frac{\left(m+\frac{1}{2}\right)\pi}{\pi-\tau}\right| > \delta, m \in \mathrm{Z}\right\},$$

where δ is sufficiently small, then there exist constant C_δ such that

$$\left|\cos(\rho(\pi-\tau))\right| \geq C_\delta e^{|t|(\pi-\tau)} > 0, \forall\lambda \in G_\delta.$$

Letting $\rho_m = m + \frac{1}{2}$ for all $m \in \mathrm{N}$ in the formula of (3.3), we find $\rho_m \in G_\delta, \cos(\rho_m\pi) = 0, \left|\cos(\rho_m(\pi-\tau))\right| \geq C_\delta > 0$, then substituting $\rho_m = m + \frac{1}{2}$ into (3.3), we arrive at

$$\pi\prod_{n=1}^{\infty}\frac{\left(m+\frac{1}{2}\right)^2 - \lambda_m^{(1)}}{m^2} = (-1)^m\left(m+\frac{1}{2}\right) + \frac{I_1\cos\left(\left(m+\frac{1}{2}\right)(\pi-\tau)\right)}{2} + O\left(\frac{1}{\rho}\right).$$

Finally, we get

$$I_1 = 2\lim_{m\to\infty}\frac{(-1)^{m-1}\left(m+\frac{1}{2}\right)+\pi\prod_{n=1}^{\infty}\frac{\left(m+\frac{1}{2}\right)^2-\lambda_m^{(1)}}{m^2}}{\cos\left(\left(m+\frac{1}{2}\right)(\pi-\tau)\right)}. \tag{3.4}$$

Lemma 3.3. Let $\pi-\tau\in\mathrm{Q}$. The spectra $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ $(j=1,2)$ of boundary spectral problems $L_j\,(j=1,2)$ uniquely determine $a_j\,(j=\ 1,2)$, respectively.

Proof. From (2.6), we have

$$a_j = \frac{2}{\pi}\lim_{n\to\infty}\left(\lambda_n^{(j)}-n^2-\frac{1}{\pi}I_1\cos(n\tau)\right)^{-1}. \tag{3.5}$$

Consequently we finish the proof.

According to Lemma 3.1-Lemma 3.4, the delay τ, the integral $I_1=\int_\tau^\pi q(t)\,dt$ and $H_j\,(j=1,2)$ are uniquely determined by the spectra $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ of $L_j\,(j=1,2)$

Next we derive the main equation of the solution of the inverse problem.

Firstly we use the function $\tilde{q}(t),K(t)$ and notations $\tilde{a}_c(\rho),\tilde{a}_s(\rho)$, $k_c(\rho),k_s(\rho)$ in [20]. Then integrating $k_c(\rho),k_s(\rho)$ by parts, we have

$$k_c(\rho)=\int_\tau^{\pi-\tau}K(t)\cos(\rho(\pi-2t))\,dt=\int_\tau^{\pi-\tau}\cos(\rho(\pi-2t))\,d\left(\int_\tau^t K(s)ds\right)$$
$$=\cos(\rho(\pi-2\tau))\int_\tau^{\pi-\tau}K(s)\,ds-2\rho k_s^*(\rho),$$

$$k_s(\rho)=\int_\tau^{\pi-\tau}K(t)\sin(\rho(\pi-2t))\,dt=\int_\tau^{\pi-\tau}\sin(\rho(\pi-2t))\,d\left(\int_\tau^t K(s)ds\right)$$
$$=-\sin(\rho(\pi-2\tau))\int_\tau^{\pi-\tau}K(s)\,ds+2\rho k_c^*(\rho),$$

Where

$$k_s^*(\rho)=\int_\tau^{\pi-\tau}\int_\tau^t K(s)\sin\left(\rho(\pi-2t)\right)dsdt,$$

$$k_c^*(\rho)=\int_\tau^{\pi-\tau}\int_\tau^t K(s)\cos\left(\rho(\pi-2t)\right)dsdt.$$

From [20] we know $\int_\tau^{\pi-\tau}K(t)\,dt=-I_2$. Thus, we get

$$\begin{aligned}
&\int_\tau^t K(s)ds=-I_2-\int_t^{\pi-\tau}K(s)ds,\\
&k_c(\rho)=-I_2\cos(\rho(\pi-2\tau))-2\rho k_s^*(\rho),k_s(\rho)=I_2\sin(\rho(\pi-2\tau))-2\rho k_c^*(\rho),\\
&k_c^*(\rho)=\int_\tau^{\pi-\tau}\int_\tau^t K(s)\cos(\rho(\pi-2t))dsdt=-\frac{I_2\sin(\rho(\pi-2\tau))}{\rho}+\tilde{k}_c(\rho),\\
&k_s^*(\rho)=\tilde{k}_s(\rho),
\end{aligned}$$

Where

$$\tilde{k}_c(\rho)=-\int_\tau^{\pi-\tau}\int_t^{\pi-\tau}K(s)\cos\left(\rho(\pi-2t)\right)dsdt,$$

$$\tilde{k}_s(\rho)=-\int_\tau^{\pi-\tau}\int_t^{\pi-\tau}K(s)\sin\left(\rho(\pi-2t)\right)dsdt.$$

Therefore (2.5), it is easy to show that the characteristic function has the formz

$$\begin{aligned}
\Delta_j(\lambda)=&H_j\left(\rho^2\right)\frac{\sin(\rho\pi)}{\rho}+\cos(\rho\pi)+\frac{1}{2\rho}\left[I_1\sin\left(\rho(\pi-\tau)\right)-\tilde{a}_s(\rho)\right]\\
&-\frac{H_j\left(\rho^2\right)}{2\rho^2}\left[I_1\cos\left(\rho(\pi-\tau)\right)-\tilde{a}_c(\rho)\right]+\frac{1}{2\rho^2}\left[I_2\cos\left(\rho(\pi-2\tau)\right)+\rho k_s^*(\rho)\right]\\
&-\frac{H_j\left(\rho^2\right)}{2\rho^3}\left[I_2\sin\left(\rho(\pi-2\tau)\right)+\rho k_c^*(\rho)\right].
\end{aligned} \tag{3.6}$$

We define function $A_j(\rho)$

$$A_j(\rho) = 2\rho^2\Delta_j(\rho^2) - 2\rho H_j(\rho^2)\sin(\rho\pi) - 2\rho^2\cos(\rho\pi) - \rho I_1\sin(\rho(\pi-\tau)) + H_j(\rho^2) I_1\cos(\rho(\pi-\tau)). \quad (3.7)$$

Function $A_j(\rho)$ is determined by $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ $(j=1,2)$, and from (3.6) and (3.7) we have

$$A_j(\rho) + \frac{H_j(\rho^2)}{\rho} I_2\sin(\rho(\pi-2\tau)) - I_2\cos(\rho(\pi-2\tau)) = H_j(\rho^2)\left[\tilde{a}_c(\rho) - k_c^*(\rho)\right] - \rho\left[\tilde{a}_s(\rho) - k_s^*(\rho)\right]. \quad (3.8)$$

Let

$$A(\rho) = A_1(\rho) + \frac{H_1(\rho^2)}{\rho} I_2\sin(\rho(\pi-2\tau)) - I_2\cos(\rho(\pi-2\tau)),$$

$$B(\rho) = A_2(\rho) - A_1(\rho) + \frac{H_1(\rho^2) + H_2(\rho^2)}{\rho} I_2\sin(\rho(\pi-2\tau)),$$

it is obvious that $A(\rho)$ and $B(\rho)$ can be uniquely determined by $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ $(j=1,2)$.

Then, from (3.8), we have

$$A(\rho) = H_1(\rho^2)\left[\tilde{a}_c(\rho) - k_c^*(\rho)\right] - \rho\left[\tilde{a}_s(\rho) - k_s^*(\rho)\right] \quad (3.9)$$

and

$$B(\rho) = \left[H_2(\rho^2) - H_1(\rho^2)\right]\left[\tilde{a}_c(\rho) - k_c^*(\rho)\right]. \quad (3.10)$$

According to (B.प) and (B.10) for $\rho = m, m \in \mathbb{Z} \square \{0\}$, we get

$$A(m) = H_1(m^2)\left[(-1)^m \int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}} \tilde{q}(t)\cos(2mt)dt - (-1)^m \int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau} K(s)\cos(2mt)dsdt\right] + m\left[(-1)^m \int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}} \tilde{q}(t)\sin(2mt)dt - (-1)^m \int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau} K(s)\sin(2mt)dsdt\right]$$

(3.11)

and

$$B(m)=\left[H_2\left(m^2\right)-H_1\left(m^2\right)\right]\times$$
$$\left[(-1)^m\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}}\tilde{q}(t)\cos(2mt)\,dt-(-1)^m\int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau}K(s)\cos(2mt)\,dsdt\right].$$

Then, from the assumption $H\left(m^2\right):=H_2\left(m^2\right)-H_1\left(m^2\right)\neq 0$, we can get

$$\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}}\tilde{q}(t)\cos(2mt)dt-\int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau}K(s)\cos(2mt)dsdt=-\frac{(-1)^m}{H\left(m^2\right)}B(m)$$

(3.12)

and

$$\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}}\tilde{q}(t)\sin(2mt)\,dt-\int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau}K(s)\sin(2mt)\,dsdt=\frac{(-1)^m}{m}A(m)-\frac{H_1\left(m^2\right)}{mH\left(m^2\right)}B(m).$$

(3.13)

Also, from (3.10) we have

$$\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}}\tilde{q}(t)dt-\int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau}K(s)\,dsdt=\frac{1}{b_1-b_2}\lim_{\rho\to 0}B(\rho). \tag{3.14}$$

We define function $K^*:[0,\pi]\to \mathrm{R}$, i.e.,

$$K^*(t)=\begin{cases}\int_{t}^{\pi-\tau}K(s)\,ds, t\in(\tau,\pi-\tau)\\ 0, t\in(0,\tau)\bigcup(\pi-\tau,\pi)\end{cases} \tag{3.15}$$

Let

$$\tilde{q}(t)-K^*(t):=g(t), t\in[0,\pi]$$

using (3.12), (3.13), and (3.14), then the Fourier series of $g(t)$ is

$$g(t)=\frac{a_0}{2}+\sum_{m=1}^{+\infty}\left[a_n\cos(2mt)+b_n\sin(2mt)\right] \tag{3.16}$$

Where

$$a_0 = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}} \tilde{q}(t)dt - \frac{2}{\pi}\int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau} K(s)\,ds\,dt = \frac{2}{\pi(b_1-b_2)}\lim_{\rho\to 0} B(\rho),$$

$$a_n = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}} \tilde{q}(t)\cos(2mt)\,dt - \frac{2}{\pi}\int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau} K(s)\,ds\cos(2mt)\,dt$$
$$= -\frac{2(-1)^m}{\pi H(m^2)} B(m), n \ge 1,$$

$$b_n = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}} \tilde{q}(t)\sin(2mt)\,dt - \frac{2}{\pi}\int_{\tau}^{\pi-\tau}\int_{t}^{\pi-\tau} K(s)\,ds\sin(2mt)\,dt$$
$$= \frac{2}{\pi}\left(\frac{(-1)^m}{m}A(m) - \frac{H_1(m^2)}{mH(m^2)}B(m)\right), n \ge 1.$$

Therefore,

$$\sum_{m=1}^{+\infty}\left[a_n\cos(2mt) + b_n\sin(2mt)\right]$$
$$= \frac{2}{\pi}\sum_{m=1}^{+\infty}\left[\frac{-(-1)^m}{H(m^2)}B(m)\frac{e^{i2mt}+e^{-i2mt}}{2}\right.$$
$$= \frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}} c_m e^{2imt},$$

where

$$c_m = -\frac{(-1)^m}{H(m^2)}B(m) + i\,\frac{(-1)^{m-1}A(m)H(m^2) + H_1(m^2)B(m)}{mH(m^2)}.$$

Since $g(t)$ is continuous function, according to Fourier series convergence theorem, we get

$$\tilde{q}(t) - \int_{t}^{\pi-\tau} K(s)\,ds = h, t \in [0,\pi] \tag{3.17}$$

where $h = \frac{1}{\pi(b_1 - b_2)} \lim_{\rho \to 0} B(\rho) + \frac{1}{\pi} \sum_{m \in Z, \{0\}} c_m e^{2imt}$.

It follows from the Lemma 3.1-Lemma 3.3 that the right-hand side of (3.17) is uniquely determined by spectra $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ of L_j, $j = 1, 2$.

Theorem 3.4. The potential $q(x)$ is uniquely determined by spectra $\left\{\lambda_n^{(j)}\right\}_{n\geq 0}$ of L_j

Proof. Since the potential $q(x)$ satisfies integral equation (3.17), we only need to show uniqueness of solution of this equation. From the defination of $K(s)$ in [20], we have $\int_t^{\pi-\tau} K(s)\,ds = 0, t \in [0, \pi] \setminus [\tau, \pi - \tau]$.

(1) For $\mathrm{t} \in \left(\pi - \tau, \pi - \frac{\tau}{2}\right]$, we have $\int_t^{\pi-\tau} K(s)\,ds = 0$. Therefore, integral equation (3.1) has a form:

$$\tilde{q}(t) = f(t)$$

The right-hand side of (3.17) is determined by $\left\{\lambda_n^{(j)}\right\}_{n\geq 0} (j = 1, 2)$ of L_j. Then from the definition of $\tilde{q}(t)$ [20], the potential $q(x)$ is determined for $x \in \left(\pi - \frac{\tau}{2}, \pi\right]$.

(2) For $\mathrm{t} \in \left(\frac{\tau}{2}, \tau\right]$, we have $\int_t^{\pi-\tau} K(s)\,ds = 0$. Therefore, integral equation (3.1) has a form:

$$\tilde{q}(t) = f(t)$$

The right-hand side of (3.17) is determined by $\left\{\lambda_n^{(j)}\right\}_{n\geq 0} (j = 1, 2)$ of L_j. Then from the definition of $\tilde{q}(t)$ [20], the potential $q(x)$ is determined for $x \in \left(\tau, \frac{3\tau}{2}\right]$.

(3) For $\mathrm{t} \in (\tau, \pi - \tau]$, according to the defination of $K(s)$, we can easily show that arguments of the potential $q(x)$ appearing in the function

$$\int_t^{\pi-\tau} K(s)\,ds = \int_t^{\pi-\tau}\left(q(s+\tau)\int_\tau^s q(u)\,du - q(s)\int_{s+\tau}^{\pi} q(u)du + \int_{s+\tau}^{\pi} q(u)q(u-s)\,du\right)ds$$

belong to the intervals $\left[2\tau,\pi\right]\subset\left[\pi-\frac{\tau}{2},\pi\right]$ and $\left[\tau,\pi-\tau\right]\subset\left[\tau,\frac{3\tau}{2}\right]$. Then the function $\int_t^{\pi-\tau} K(s)\,ds$ is known. Therefore from (3.]) for $t\in\left(\tau,\pi-\tau\right]$. Then from the definition of $\tilde{q}(t)$[20], the potential $q(x)$ is determined for $x\in\left(\frac{3\tau}{2},\pi-\frac{\tau}{2}\right]$.

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